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R. Gatto: THE TWO NEUTRINOS.

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INTRODUCTION

I shall talk essentially on the following subjects

1) High-energy neutrinos:

   1a) Elastic cross-sections in the simplest hypothesis.
   1b) Elastic cross-sections in more general hypotheses.
   1c) Cross-sections on complex nuclei.
   1d) Inelastic cross-sections.

2) μ - e symmetry and its consequences, namely:

   2a) Two neutrinos.
   2b) A multiplicative μ - e selection rule.

3) Additive selection rule: additive versus multiplicative selection rule.

4) Further symmetries:

   4a) Pauli-Gursey transformation.
   4b) SU(2) or else (?).

HIGH-ENERGY NEUTRINOS

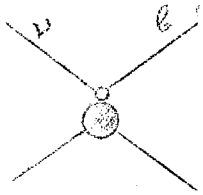
Elastic cross-sections in the simplet hypothesis

1. Assuming lepton conservation, the two elastic reactions (the nomenclature elastic is here only used for convenience) are

(1) $$\bar{\nu} + p \to \ell^+ + n$$

$$\nu + n \to \ell^- + p$$

We describe a reaction such as (1) in terms of a graph of the kind



The $\nu - \ell$ pair is coupled at a single space-time point through a current

(3) $$\ell_\mu = (\bar{\nu}\, \gamma_\mu\, a\, \ell)$$

where $a = \frac{1}{2}(1 + \gamma_5)$, and the other notations are obvious. The matrix element for (1) is given by

(4) $$\sqrt{2}\, G \frac{i}{(2\pi)^2}\, \delta_4(\bar{\nu} + p - \ell^+ - n) j_\mu \ell_\mu$$

where G is the weak coupling constant

(5) $$G \cong 10^{-5}\, M^{-2}$$

(M = nucleon mass), $\bar{\nu}$, p, ℓ^+, n are used also to denote the four-momenta, and

(6) $$j_\mu = (2\pi)^3 \langle n | j_\mu^{(V)}(0) + j_\mu^{(A)}(0) | p \rangle$$

We have split the weak (strangeness-conserving) current into a sum of a vector part and an axial part.

2. A simplest set of hypothesis to determine j_μ is provided by
a) non-renormalization hypothesis for the vector current[1], i.e.

(7) $$\frac{\partial}{\partial x_\mu} j_\mu^{(V)}(x) = 0$$

b) $j_\mu^{(A)}$ behaves as the appropriate component of an isotopic spin vector.

I shall illustrate later the meaning of such hypotheses. Under a) and b) (and, of course, also assuming invariance under the Lorentz group including time reversal) one can write a simple form for j_μ, valid at sufficiently high energies such that the mass of ℓ can be neglected:

(8) $$j_\mu = u(\bar{n}) \left[H_1(K^2) \gamma_\mu \gamma_5 + F_1(K^2) \gamma_\mu + \frac{\mu}{2M} F_2(K^2) \sigma_{\mu\nu} K_\nu \right] u(p)$$

In Eq. (8), $u(\bar{n})$ and $u(p)$ are Dirac spinors, $H_1(K^2)$, $F_1(K^2)$ and $F_2(K^2)$ are form factors depending on the invariant momentum transform K^2, where

(9) $$K_\mu = (p - n)_\mu$$

They are normalized according to

(10) $$H_1(0) = \frac{G_A}{G} = 1.2 \qquad F_1(0) = F_2(0) = 1$$

and, as it directly follows from a), F_1 and F_2 are differences of the corresponding proton and neutron elettromagnetic form factors. Also $\mu = \mu_p - \mu_n \cong 3.7$ in nucleon Bohr magnetons. In the limit $K^2 \to 0$

(11) $$j_\mu \to \bar{n} \gamma_\mu (1 + \frac{G_A}{G} \gamma_5) p$$

The differential corss-section [2] derived from Eqs. (4), (3), (6) and (8) is reported in Appendix 1.

3. What do we expect in general for the behaviour of the cross-section? If E is centre-of-mass energy, then phase space goes as E^2 and the amplitude is proportional to G. On this groung one would have

(12) $$\sigma \propto E^2 G^2 = \frac{1}{M^2} \left(\frac{E}{M} \right)^2 10^{-10} \propto \mathcal{E}_\nu G^2$$

However there are form factors present. What do they do? Suppose they are essentially equivalent to a cut-off at $K^2 = a^2$, in the sense that momentum transfers $> a^2$ are excluded. Then we have to multiply Eq. (12) by a factor expressing the "allowed" solid angle. But $K^2 < a^2$ means

(13) $$4 |\vec{p}|^2 \sin^2 \frac{\theta}{2} < a^2$$

where $\vec{p}$ is the c.m. proton momentum. Therefore the maximum allowed solid angle is

(14) $$\Delta\omega = 4\pi \sin^2 \frac{\theta_{max}}{2} \simeq \pi \frac{a^2}{|\vec{p}|^2}$$

and for large energies $\Delta\omega \propto E^{-2}$, so that

(15) $$\sigma \propto E^2 G^2 \Delta\omega \rightarrow G^2 a^2$$

in the high-energy limit, i.e. it goes to a constant. To elaborate a little better on this point let me write down the expression that you get for σ_{total} from the expression in Appendix I, in the high-energy limit. You find, in the high-energy limit, for both (1) and (2)

(16) $$\sigma_{tot} \rightarrow G^2(h_1 + f_1 + f_2)$$

where

(17) $$h_1 = \frac{1}{2\pi} \int_0^\infty d\, K^2 H_1{}^2(K^2)$$

(17') $$f_1 = \frac{1}{2\pi} \int_0^\infty d\, K^2 F_1{}^2(K^2)$$

(17") $$f_2 = \frac{1}{2\pi} \left(\frac{\mu}{2M}\right)^2 \int_0^\infty K^2\, d\, K^2 F_2{}^2(K^2)$$

If we now choose for the form factors

(18) $$F_1(K^2) = \frac{1}{\left(1 + \frac{K^2}{a_1^2}\right)^2}$$

(18') $$F_2(K^2) = \frac{1}{\left(1 + \frac{K^2}{a_2^2}\right)^2}$$

(18") $$H_1(K^2) = \frac{G_A}{G_V} \frac{1}{\left(1 + \frac{K^2}{b^2}\right)^2}$$

we find, in the high-energy limit,

(19) $$\sigma_{total} = \frac{1}{6\pi}\left[G^2 a_1^{\ 2} + \frac{\mu^2}{8M^2} G^2 a_2^{\ 2} + G_A^{\ 2} b^2 \right]$$

which is precisely of the form (15). If you put $a_1^{\ 2} = a_2^{\ 2} = b^2 = 37.5\ m_\pi^2$ (as was suggested by the Stanford experiments) you find

(20) $$\sigma_{total} \cong 0.75 \times 10^{-38}\ cm^2$$

The statement (15) that $\sigma_{tot} \to$ constant is only to be intended in a practical sense. For instance, if the form factors are written in the more fashionable form

$$F = (1 - V) + \frac{V}{1 + \frac{K^2}{\rho^2}}$$

where ρ is some mass, then according to Eq. (16)

$$\sigma \to \infty$$

If they are written as

$$F = \frac{a}{1 + \frac{K^2}{\rho^2}} + \frac{b}{1 + \frac{K^2}{x^2}}$$

where x is another mass, then σ increases logarithmically. And so on.

6. In the high-energy limit the differential cross-section of Appendix I takes a simple form. Always in the laboratory frame call $\mathcal{E}_\nu$, $\mathcal{E}_\ell$ the energies of and ℓ, and T the kinetic energy of the recoil proton. Then

(21) $$K^2 = \frac{\mathcal{E}_\nu - \mathcal{E}_\ell}{2M} = \frac{T}{2M}$$

and introduce

(22) $$x^2 = \frac{K^2}{(2M)^2}$$

Then, for both (1) and (2), in the high-energy limit

(23) $$\frac{d\sigma}{d\mathcal{E}_\ell} \to \frac{1}{\pi} MG^2 (H_1^{\ 2} + E_1^{\ 2} + \mu^2 x^2 F_2^{\ 2})$$

where the form factors are taken at K^2.

7. It may be instructive also to talk of the polarization. In a scattering processes you expect a polarization normal to the scattering plane even starting by unpolarized particles. However, theamplitude in our case is real (we are assuming time reversal invariance) and there is no such polarization normal to the plane. The recoil nucleon can, however, have a longitudinal polarization, just originating from the fact that the processes is parity non-conserving. For reaction (1), under the simplest hypothesis a) and b) described above, the longitudinal polarization of the final neutron becomes

$$(24) \qquad P_L \to \frac{x}{\sqrt{1+x^2}} \quad \frac{2H_1(F_1 + \mu F_2)}{H_1{}^2 + F_1{}^2 + \mu^2 F_2{}^2 x^2}$$

For reaction (2) the final proton has a longitudinal polarization given by $-P_L$. Under the rough assumption we have already used, of all form factors equal, you get

$$(25) \qquad P_L \to \frac{x}{\sqrt{1+x^2}} \quad \frac{(4.5)}{1 + (5,6)\, x^2}$$

Elastic cross-sections in more general hypothesis.

8. If I relax the hypothesis a) and b) and rely only on Lorentz invariance (and time reversal) I must write for the general form of j_μ

$$(26) \qquad \begin{aligned} j_\mu = \bar{n} \Big[& F_1 \gamma_\mu + \frac{\mu}{2M} F_2 \sigma_{\mu\nu} K_\nu + iF_3 K_\mu + \\ & + H_2 \gamma_\mu \gamma_5 + \frac{\mu'}{2M} H_3 \sigma_{\mu\nu} \gamma_5 K_\nu + iH_2 K_\mu \gamma_5 \Big] p \end{aligned}$$

where the f's are form factors for the vector part and the h's are form factors for the axial part. Equation (26) is a rigorous but perhaps too general framework for analysing the experiments. Thus, next to the simplest hypothesis for j_μ discussed before, we will consider a possible situation where a) (i.e. non-renormalization for the vector part) still holds, but b) (i.e. behaviour of $j_\mu{}^{(A)}$ as a component of an isospin vector) does not necessarily hold. In fact, one has a theoretical feeling that the non-renormalization hypothesis for the vector strangeness-conserving current is true. By the way, this implies that such a current transforms as a vector in isospin space (it is actually assumed to be the isospin current itself). On the other hand, the behaviour of the axial strangeness-conserving current under isospin rotations has to be postulated. It is true that in a possible scheme where you first construct the vector current as the isospin current out of the known particles, and then you generate the axial current by inserting a γ_5 for each fermion pair, you get a current that transforms as a vector in isospin space. However, everybody would feel that this procedure is too much model-dependent to be completely reliable. Thus, although we like the hypothesis b), it may be opportune to allow for its possible non-validity. Now from a) you have directly

(27) $K_\mu j_\mu = 0$ or equivalently $F_3 = 0$

Furthermore you can safely neglect the term proportional to H_2 since

(28) $K_\mu \ell_\mu \propto m_e$

so you can neglect it at high energies, assuming H_2 does not become pathologically big. In conclusion, the step next to the simplest hypothesis, discussed in 1a), will be to assume

$$j_\mu = \bar{n}\left[F_1\gamma_\mu + \frac{\mu}{2M}F_2\sigma_{\mu\nu}K_\nu + H_1\gamma_\mu\gamma_5 + \frac{\mu'}{2M}H_3\sigma_{\mu\nu}\gamma_5 K_\nu\right]p \tag{29}$$

where F_1 and F_2 are still differences of the corresponding proton and neutron electromagnetic form factors.

9. The new term is that containing $\sigma_{\mu\nu}K_\nu\gamma_5$. The high-energy limit (16) now becomes

(29) $\sigma_{tot} \to G^2(h_1 + h_3 + f_1 + f_2)$

where h_1, f_1 and f_2 are still given by Eqs. (17), (17') and (17"), and

(30) $h_3 = \frac{1}{2M}\left(\frac{\mu'}{2M}\right)^2\int_0^\infty K^2dK^2H_3{}^2(K^2)$

Similarly, Eq. (23) becomes

(31) $\frac{d\sigma}{d\mathcal{E}_\ell} \to \frac{1}{\pi}MG^2\left[H_1{}^2 + F_1{}^2 + (\mu^2F_2{}^2 + \mu'^2H_3{}^2)\;x^2\right]$

and Eq. (24) becomes

$$(32) \quad P_L \to \frac{x}{\sqrt{1+x^2}} \; \frac{2\left[(H_1 - \mu' H_3)(F_1 + \mu F_2) + \mu\mu' F_2 H_3 (1+x^2)\right]}{H_1^{\,2} + F_1^{\,2} + (\mu^2 F_2^{\,2} + \mu'^2 H_3)\, x^2}$$

Cross-sections on complex nuclei

10. We shall briefly deal here with the effect of the Pauli principle on the cross-section when a reaction such as (1) or (2) occurs on a complex nuclei. Consider, for instance, reaction (1)

$$(1) \qquad \nu + p \to \ell^+ + n$$

The individual protons of the nucleus act incoherently so that at large momentum transfers the cross-section on the nucleus is Z times the cross-section on a proton. However, in an independent particle language at small momentum transfers the recoil n may not find a free final state where to jump and the reaction may therefore become forbidden. Let us see roughly what happens. In the laboratory system the momentum transfer K^2 is given by

$$(33) \qquad K^2 = (p - n)^2 = |\vec{n}|^2 - (\mathcal{E}_n - M)^2 = 2MT$$

where we have denoted with n and $\mathcal{E}_n$ the laboratory neutron momentum and total energy (T is the neutron kinetic energy). Now consider a Fermi model. In order for the reaction to occur we can say that T must be larger than the Fermi energy T_F so that the neutron can escape from the Fermi sphere. This implies

$$(34) \qquad K^2 = 2MT > 2MT_F = 2M \frac{p_F^2}{2M} = p_F^2$$

where p_F is the Fermi momentum. Still in the laboratory system

$$(35) \qquad K^2 = (p - n)^2 = (\ell - \nu)^2 \cong 4\, \mathcal{E}_\ell \mathcal{E}_\nu \sin^2 \frac{\theta}{2}$$

neglecting the mass of ℓ. From Eqs. (35) and (34)

$$4\, \mathcal{E}_\ell \mathcal{E}_\nu \sin^2 \frac{\theta}{2} > p_F^2$$

$$(36) \qquad 2 \sin \frac{\theta}{2} > \frac{p_F}{\sqrt{\mathcal{E}_\ell \mathcal{E}_\nu}}$$

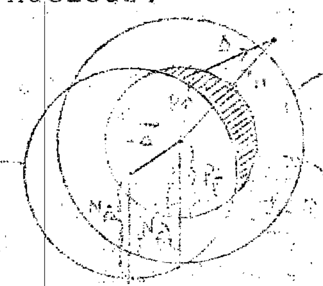
Since $\mathcal{E}_\nu - T = \mathcal{E}_\ell$, we can neglect T (here always $< T_F$) and write Eq. (36) in the form

(37) $$\Theta > \sim \frac{p_F}{\mathcal{E}_\nu}$$

If $\mathcal{E}_\nu \cong 0.7$ BeV and $p_F = 0.2$ BeV you find $\Theta > \sim 10°$. Reactions occurring at an angle Θ less than $\sim p_F/\mathcal{E}_\nu$ are thus cut off from the exclusion principle. For a rough picture of the process on a complex nucleus we have so to take into account two cut-off angles. One for small angles, less than $\sim p_F/\mathcal{E}_\nu$ due to the Pauli principle, the other for large angles greater than Θ_{max} (see Eq.14) due to the nucleon form factors.

11. A better estimate can be made, as usual in similar cases, by calculating the fraction of volume of the proton Fermi sphere that contains those protons which, for a given momentum transfer $\vec{\Delta} = \vec{n} - \vec{p}$, change into neutrons outside the neutron Fermi sphere. Then considerder the proton Fermi sphere of radius P_F (fig. 1)

Fig.1 - Fermi sphere description of ν scattering on a nucleus.-



Draw the neutron Fermi sphere of radius N_F with a centre displaced by $\vec{\Delta}$ from the centre of the proton sphere. A proton lying in the shaded volume satisfies the condition that $|\vec{p}+\vec{\Delta}| > N_F$. The fraction of the "effective" protons is then given by the ratio of the volume of this region to the total volume. Defining

(38) $$\eta = \frac{\Delta}{N_F} \qquad \eta_0 = 1 - \frac{P_F}{N_F}$$

one finds [3] for the fraction δ_Z of the "effective" protons at a momentum transfer Δ

(39) $$\delta_Z = 0 \qquad \text{for } \eta < \eta_0$$

(39') $\quad \delta_Z = \frac{N}{Z} F(\eta) - \left(\frac{N}{Z} - 1\right)$, for $\eta > \eta_o$

until, at large Δ, $\delta_Z = 1$. In Eq. (39'), $F(\eta)$ is given by

$$F(\eta) = \frac{3}{4}\eta - \frac{1}{16}\eta^3 + \frac{1}{2}\left(1 - \frac{Z}{N}\right) +$$

$$+ \frac{3}{16\eta}\left[1 - \left(\frac{Z}{N}\right)^{2/3}\right]\left[1 - \left(\frac{Z}{N}\right)^{2/3} - 2\eta^2\right]$$

Similarly, for reaction (2)

(2) $\quad \nu + n \rightarrow \ell^- + p$

the fraction δ_N of the "effective" neutrons is given by

(40) $\quad \delta_N = 1 - \frac{Z}{N}$ for $\eta < \eta_o$

(40') $\quad \delta_N = F(\eta)$ for $\eta > \eta_o$

until it also reaches the value one. With this more accurate treatment you get, instead of the rigorous cut-off for $\theta < p_F/E_\nu$, a smooth depression for the small angles that satisfy the above inequality.

Inelastic cross-sections

12. Some of the inelastic cross-sections are still two-body reactions

(41) $\quad \bar{\nu} + p \rightarrow \ell^+ + \Lambda^o$

(42) $\quad \bar{\nu} + p \rightarrow \ell^+ + \Sigma^o$

(43) $\quad \bar{\nu} + n \rightarrow \ell^+ + \Sigma^-$

(44) $\quad \bar{\nu} + n \rightarrow \ell^- + \Sigma^+$

and similarly starting from ν. Note that the last reaction (44) goes through a current with $\Delta S = -\Delta Q$, which for a long time has been assumed not to exist. But now there seems to be good experimental evidence in favour of such a current.

Other inelastic processes are

(45) $\bar{\nu} + p \to \ell^+ + n + \pi^0$

(46) $\bar{\nu} + p \to \ell^+ + p + \pi^-$

(47) $\bar{\nu} + n \to \ell^+ + n + \pi^-$

and similar processes starting from ν.

One has also processes

(48) $\bar{\nu} + p \to \ell^+ + p + K^-$

(49) $\bar{\nu} + p \to \ell^+ + n + K^0$

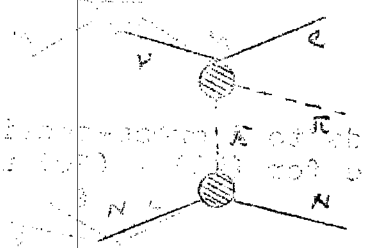
(50) $\bar{\nu} + p \to \ell^+ + n + \bar{K}^0$

and similarly from ν.

13. The reactions (41) - (44) are inverse hyperon β-decay processes. One knows that their experimental rates are lower than universal Fermi theory would predict. One this basis one would also expect that (41) - (44) have smaller cross-sections (say, by a factor of 10 or more) than the elastic cross-sections we considered before. However, this statement is far from being rigorous. The momentum transfer K^2 equal, say, to $(Y - p)^2$, takes on generally different values in the decay and in the high-energy production processes. Knowledge of the decay only gives very partial information on the production.

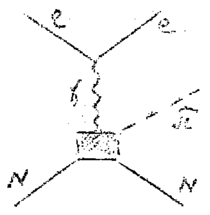
14. As to reactions (45) - (47) [4] estimates have been given for the vector part by

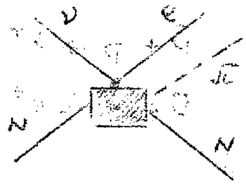
i) estimating the contribution from the one-pion pole



In this graph the vertex ($\pi \pi e \nu$) is expressed in terms of the electromagnetic pion form factor (on which theorists think they already know a lot) by use of the non-renormalization hypothesis. For an energy $E_\nu \sim 1$ BeV one finds $\sigma \sim 10^{-39}$ cm^2.

ii) Comparing with electropion production, as shown by the graphs



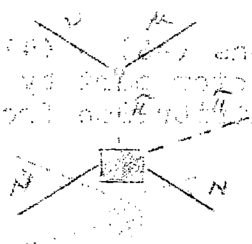


Again the connection is provided by the non-renormalization hypothesis. One would roughly have

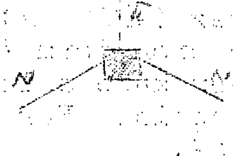
$$(51)\quad \left(\frac{d\sigma}{d\Omega_\ell\, dE_\ell}\right)_{\text{neutrinopion}} = \frac{1}{4}\left(\frac{GK^2}{4\pi\alpha}\right)^2\left(\frac{d\sigma}{d\Omega_\ell\, dE_\ell}\right)_{\text{electropion}}$$

where K is the momentum transfer and $\alpha = 1/137$. Extrapolating existing electropion data at a lower energy one finds in this way a cross-section of the order of $10^{-39}\ cm^2$.

As to the axial part, an estimate can be made of the onepion pole (when the final lepton is a muon, otherwise this pole is negligible). The graph is



The estimate leads to a cross-section $\sim 10^{-40}\ cm^2$. A one--pion approximate for (48) - (50) uses the graph



and therefore connects the process to the decay

$$K \to \pi + \mu + \nu$$

The cross-section is again small $\sim 10^{-40}$ cm^2.[5)]

μ-e Symmetry and its consequences

15. In this section I shall describe an approach that we proposed about two years ago which led us to postulate a property of μ-e symmetry[6)]. From such a μ-e symmetry it followed in a natural way that

i) there are two neutrinos;

ii) there is a multiplicative selection rule forbidding transformations of μ into e.

Two neutrinos and the existence of a selection rule are now verified experimentally. The selection rule may be an additive, instead of multiplicative selection rule. We shall discuss later in this section the experimental differences between the two kinds of selection rules. The formal argument that we shall give runs roughly as follows. Let us assume that μ and e have identical interactions and they only differ in their rest mass. It can then be seen that, if electromagnetic interactions satisfy the requirement of minimality and if weak interactions are neglected, there exists a symmetry property of the theory, that we call μ-e symmetry, related to an exchange of the two lepton fields that are necessary to describe μ and e. We then find that in order to maintain the μ-e symmetry also in a complete theory including weak interactions one has to postulate two neutrinos. This possibility of having two neutrinos has been considered since a long time by different authors on various grounds. Experimentally it seems to be suggested from the absence $\mu \to e + \gamma$, which should almost certainly be present if in $\mu \to e + \nu + \bar{\nu}$ there is only one kind of neutrino. Now, it is known that to a symmetry property of the theory there corresponds, in general, a physical conservation law. The conservation law corresponding to μ-e symmetry is a multiplicative conservation law that forbids transformations of μ into e.

More specifically, one assigns to each particle a multiplicative muonic quantum number K, according, for instance, to the assignment shown in Table I, and one obtains that the only reactions that are permitted are those for which the product of the initial K values is equal to the product of the final K values. To the mesons, baryons and photon we have assigned K = 1, but we could have chosen instead K = exp $[i(\pi/2)N]$ where N is the nucleonic quantum number of the particle, without altering any physical consequence, but only with a slight altera-

Table I

particles	quantum number K	quantum number M
μ^-, ν_μ, e^+, $\bar{\nu}_e$	$-i$	-1
μ^+, $\bar{\nu}_\mu$, e^-, ν_e	$+i$	$+1$
mesons, baryons, γ	1	0

tion of the formal argument leading to the postulated law. In Table I there is also a column defining the quantum number M. This is the additive quantum number and we shall talk of it later.

16. Let us now develop the formal argument in some more detail. If we neglect weak interactions the total Lagrangian looks like

$$(52) \qquad \mathcal{L} = -\bar{e}(\gamma\partial + m_e)e - \bar{\mu}(\gamma\partial + m_\mu)\mu + \text{other terms.}$$

The "other terms" do not contain e or μ. The electromagnetic interaction of μ and e is included in Eq. (52)--following our requirement of minimality--through the definition of ∂

$$(53) \qquad \partial = \frac{\partial}{\partial x} + ieA.$$

The way it is written, Eq. (52) is not symmetric under the exchange $\mu \leftrightarrow e$. However, nobody told us what to call a μ and what to call an e. If I introduce new fields e' and μ' through

$$(54) \qquad e = \frac{1}{\sqrt{2}}(e' + \mu')$$

$$(54') \qquad \mu = \frac{1}{\sqrt{2}}(e' - \mu')$$

$\mathcal{L}$ takes the form

$$\mathcal{L} = -\bar{e}'(\gamma\partial + m_+)e' - \bar{\mu}'(\gamma\partial + m_+)\mu' +$$

$$(55) \qquad + m_- \left[(\bar{e}'\mu') + (\bar{\mu}'e') \right] + \text{other terms.}$$

In this form $\mathcal{L}$ is symmetric under $e' \leftrightarrow \mu'$.

17. If somebody is sceptical about this completely formal argument one can point out that the situation here is not really quite different from what one does with the "physical" particles K_1^o and K_2^o. For reasons of symmetry one prefers to use in strong interactions K^o and $\bar{K}^o$ that are related by charge conjugation. At some point there is CP conservation, however, that tells you that the "physical" particles are K_1^o and K_2^o. Similarly the hope here is that at some point there will be a conservation law (it will be muonic number conservation) that will tell you that the "physical" particles are just μ and e.

18. It is convenient now to introduce a fictious L space (lepton space). In L space, e and μ form a doublet.

(56) $$\psi = \begin{bmatrix} e \\ \mu \end{bmatrix}$$

The transformation (54) and (54') can then be generalized to

(57) $$\psi = T^{-1} \psi'$$

where T is a non-singular matrix. Then $\mathcal{L}$ takes the general form

(58) $$\mathcal{L} = - \bar{\psi}' \left[\gamma\partial (A+B\gamma_5) + (C+iD\gamma_5) \right. + \text{other terms.}$$

In Eq. (58) A, B, C and D are hermitian matrices in L space (i.e. they are 2 x 2 matrices acting on the spinors ψ), and furthermore A+B and A-B are both positive definite. The matrix T can be decomposed as

(59) $$T = aR + \bar{a}S$$

where $a = \frac{1}{2}(1+\gamma_5)$, $\bar{a} = \frac{1}{2}(1-\gamma_5)$, and R and S are matrices in L space. If we define the mass matrix M as

(60) $$M = m_+ + m_- \sigma_3$$

the conditions that R and S must satisfy to bring Eq. (52) into Eq. (58) are

(61) $R^{\dagger}\ (A+B)R = 1$

(61') $S^{\dagger}\ (A-B)S = 1$

(61") $S^{\dagger}\ (C+iD)R = M$

This is shown in Appendix II.

19. Now μ-e symmetry implies that Eq. (58) be invariant when

(62) $\psi' \rightarrow \sigma_1 \psi'$

as this amounts to the substitution of the two components of the doublet ($\psi_1' \leftrightarrow \psi_2'$). This condition requires that A, B, C, D all commute with σ_1, so that each of them can be written in a form

(63) $\alpha P_+ + \beta P_-$

where α and β are coefficients and

(64) $P_{\pm} = \frac{1}{2}(1 \pm \sigma_1)$

In fact Eq. (63) is the most general function of the unit matrix 1 and σ_1. There are infinite choices of R and S that satisfy Eqs. (61), (61') and (62") with A, B, C, D of the form (63).

20. So far we have neglected weak interactions. Now we require that μ-e symmetry as expressed by Eq. (62) be valid also when weak interactions are present. We shall see that this is impossible if there is only one neutrino.

A term of weak interactions is

(65) $G\left[\bar{\nu}\gamma_\mu(ae + a\mu)\right] J_\mu$

where J_μ is a current not containing leptons, but only baryons and mesons. In doublet notation Eq. (59) is

(65') $G(\bar{\nu}\gamma_\mu a\psi)\ J_\mu$

In terms of ψ'

(66) $$G(\bar{\nu}\gamma_\mu a\psi)J_\mu = G(\bar{\nu}\gamma_\mu aT^{-1}\psi')$$

and if we want μ-e symmetry

(67) $$(66) = G(\bar{\nu}\gamma_\mu aT^{-1}\sigma_1\psi') = G(\bar{\nu}\gamma_\mu aT^{-1}\sigma_1 T\psi)$$

So we must have

(68) $$a\psi = aT^{-1}\sigma_1 T\psi$$

In Appendix III it is shown that Eq. (68) is incompatible with Eqs. (61), (61') and (61").

21. Thus we have to introduce two neutrinos. Then the Lagrangian

$$\mathcal{L} = -\psi'(\gamma\partial + m_+)\psi' - m_-\bar{\psi}'\sigma_1\psi' - \bar{\nu}'\gamma\partial\nu' +$$

(69) $$+ G\left[\bar{\psi}'\gamma a\nu' + \dots\right]\left[\bar{\nu}'\gamma a\psi' + \dots\right] +$$

$$+ \text{other terms}$$

where

(70) $$\nu' = \begin{bmatrix} \nu'_e \\ \nu'_\mu \end{bmatrix}$$

is also a doublet in L space, and is now invariant under μ-e symmetry defined by

(71) $$\psi \to \sigma_1\psi \qquad \nu \to \sigma_1\nu$$

In fact Eq. (69) is the Lagrangian one usually adopts. However, Eq. (69) is not the most general Lagrangian compatible with Eq. (71).
To end up with Eq. (69) there must be more in the story. But first let us go back to the μ-e selection rule ensuing from invariance under Eq. (71). To develop a physical feeling for such a selection rule we want to know what

Eq. (71) means in terms of the good old e and μ. Now

$$\psi' \rightarrow \sigma_1 \psi'$$

means

$$T\psi \rightarrow \sigma_1 T\psi$$

or

$$\psi \rightarrow T^{-1}\sigma_1 T\psi \tag{72}$$

What is $T^{-1}\sigma_1 T$? It must be: traceless; of unit square; commute with σ_3 (because it must commute with M). So it is $\pm\ \sigma_3$. And then it is obvious which is the selection rule: it is a multiplicative [corresponding to the discreteness of the operation (71)] selection rule that forbids a μ to go into an e. How this selection rule operates we have already illustrated before.

22. Now let us state that there is a reason to believe that this frame work of μ-e symmetry, although it is sufficient to imply

i) two neutrinos

ii) a selection rule,

is perhaps too wide. First, the selection rule may very well be additive (see below). Secondly, take an example: add to the weak current in Eq. (69) the term $\lambda\overline{\Psi}\gamma_a \sigma_1 \nu'$, allowed by Eq. (71). Then, in the μ-e representation, μ is coupled proportional to $(1 - \lambda)$, and e proportional to $(1 + \lambda)$. But, as we know, all the evidence is instead for equal couplings. So we learn that there must be more to the story if one wants to get finally to Eq. (69).

Additive selection rule (Additive vs multiplicative selection rule).

23. Besides the multiplicative conservation law that we have discussed one can consider the more stringent additive conservation law: to each particle there corresponds an additive muonic quantum number M, such that, for istance,

$$K = \exp\left[i\frac{\pi}{2}M\right],$$

and only those reactions are allowed for which the sum of the initial M values is equal to the sum of the final M values. Corresponding to our previous assignments of K one can assign values of M as in Table 1.

It is obvious that if the additive muonic conservation law is verified, the multiplicative law is also verified.

However, if the multiplicative muonic conservation law is verified it does not follow that the additive conservation law is also verified. It only follows that the difference M between the sum of the initial M values and the sum of the final M values can only take the values 0, ± 4, ± 8 (i.e. $\Delta M = 0$ mod. 4).

Odd M are always forbidden because of lepton conservation. $\Delta M = \pm 2$, ± 6, etc., are forbidden for both types of conservation laws, and this is sufficient to prevent reactions such as

$$\mu \rightarrow e + \gamma$$

$$\mu \rightarrow e + e + e$$

$$\mu + (\text{nucleus}) \rightarrow e + (\text{nucleus})$$

$$\mu \rightarrow e + \nu_e + \bar{\nu}_e, \quad \pi \rightarrow \mu + \nu_e, \text{ etc.}$$

Reactions with $\Delta M = \pm 4$, ± 8, etc., are forbidden by the additive law, but they are allowed by the multiplicative law. Thus evidence for reactions like, for instance,

(73) $$e^+ + \mu^- \rightarrow e^- + \mu^+$$

(74) $$e^+ + e^+ \rightarrow \mu^+ + \mu^+$$

(75) $$\nu_\mu + (\text{nucleus}) \rightarrow \nu_e + e^- + \mu^+ + (\text{nucleus}),$$

would exclude the additive conservation law and be consistent with the multiplicative conservation law. None of these reactions can occur by electromagnetic interactions alone, and thus any evidence for them would be an evidence for some new mechanism, for instance of the kind we are considering. The reaction (73) is very interesting: it can occur as a charge exchange reaction in muonium.

$$\underset{\text{bound}}{(e^- + \mu^+)} \rightleftarrows \underset{\text{bound}}{(e^+ + \mu^-)}$$

It has been studied theoretically by Pontecorvo[7], and by Feinberg and Weinberg[8]. We refer to the work of these authors for further discussion.

The reaction (74) might, in principle, be studied by colliding beam experiments of the type considered at Stanford, but carried out at a much higher energy or with higher intensity. A possible interaction of the type

$$\mathcal{L}' = f\left(\bar{\psi}^{(\mu)}\gamma_\lambda a \psi^{(e)}\right)\left(\bar{\psi}^{(\mu)}\gamma_\lambda a \psi^{(e)}\right)$$

where

$$a = \tfrac{1}{2}(1 + \gamma_5) \text{ and } \bar{a} = \tfrac{1}{2}(1 - \gamma_5),$$

through Fierz re-ordering can be written in the form (we use the Majorana representation)

$$\mathcal{L}' = 2f\left(\psi^{(e)}\ \gamma_4\bar{a}\ \psi^{(e)}\right)\left(\psi^{(\mu)*}\ \gamma_4\bar{a}\ \psi^{(\mu)*}\right)$$

The cross-section is then clearly isotropic in the c.m. system. The total cross-section is given by

$$\sigma_{total} = \frac{f^2E^2}{\pi}\ \beta(1+\beta^2)$$

where E is the c.m. energy of each colliding electron and β is the velocity in the c.m. system. If, tentatively, we identify f with $\sqrt{8}G$, where G is the weak coupling constant, the cross-section turns out to be

$$\sigma_{total} = 1.6 \cdot 10^{-37}(E/M)^2\ cm^2$$

where M is the nucleon mass.

The final muons are polarized longitudinally. The value of the polarization is

$$P^{\pm} = \pm\frac{2\beta}{1+\beta^2}$$

where the upper sign holds for μ^+, the lower sign for μ^-.

The reaction (75) can occur through an interaction of the type

$$\mathcal{L}' = f\left(\bar{\psi}^{(\mu)}\gamma_\lambda\, a\, \psi^{(\nu_e)}\right)\left(\bar{\psi}^{(\nu_\mu)}\gamma_\lambda\, a\, \psi^{(e)}\right)$$

together with the absorption of a virtual nuclear γ ray by one of the final charged leptons. An estimate of its cross-section on lead gives $0.5 \cdot 10^{-39}$ (E_ν in GeV) cm^2 where E_ν is the incident neutrino energy in the laboratory system. The rapid increase of the cross-section with energy will perhaps make this process one of the most convenient to decide between the two possibilities of an additive or of a multiplicative muon-electron selection rule.

24. The formal expression of lepton conservation by a gauge invariance property (quite similar to that which expresses conservation of charge) is, in terms of the fields $e(x)$, $\mu(x)$, $\nu_e(x)$, $\nu_\mu(x)$

$$\begin{array}{ll} e \to e^{i\lambda}\, e & \bar{e} \to e^{-i\lambda}\, \bar{e} \\ \mu \to e^{i\lambda}\, \mu & \bar{\mu} \to e^{-i\lambda}\, \bar{\mu} \\ \nu_e \to e^{i\lambda}\, \nu_e & \bar{\nu}_e \to e^{-i\lambda}\, \bar{\nu}_e \\ \nu_\mu \to e^{i\lambda}\, \nu_\mu & \bar{\nu}_\mu \to e^{-i\lambda}\, \bar{\nu}_\mu \end{array} \tag{76}$$

with the same real number λ for all fields. Similarly if we want a continous gauge also for muonic conservation (implying the additive selection rule), muonic conservation is expressed by

$$\begin{array}{ll} e \to e^{-im}\, e & \bar{e} \to e^{im}\, \bar{e} \\ \mu \to e^{im}\, \mu & \bar{\mu} \to e^{-im}\, \bar{\mu} \\ \nu_e \to e^{-im}\, \nu_e & \bar{\nu}_e \to e^{im}\, \bar{\nu}_e \\ \nu_\mu \to e^{im}\, \nu_\mu & \bar{\nu}_\mu \to e^{-im}\, \bar{\nu}_\mu \end{array}$$

25. We now write Eqs. (76) and (77) in a formalism in which the two 2-component spinors ν_e and ν_μ are joined to form a single 4-component spinor. In such a formalism you can still go on saying that there is only one neutrinO.

I shall work in a representation with γ_5 diagonal. If you take

$$\gamma_K = \begin{bmatrix} 0 & -i\sigma_K \\ i\sigma_K & 0 \end{bmatrix} \quad \gamma_4 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \gamma_5 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

(78)

$$\left(\text{with } \sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right),$$

you find that the γ's are hermitian, satisfy $\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\delta_{\mu\nu}$, and $\gamma_5 = \gamma_1 \gamma_2 \gamma_3 \gamma_4$ (In fact, the representation you are accustomed to is one in which one has simply replaced $\gamma_4 \to -\gamma_5$, $\gamma_5 \to \gamma_4$, leaving the commutation relations and $\gamma_5 \gamma_4 = \gamma_1 \gamma_2 \gamma_3$ unchanged). The equation for a two-component left-handed neutrino is obtained from

(79) $$\gamma \partial \psi = 0$$

by projecting out with

(80) $$\tfrac{1}{2}(1 - \gamma_5)\psi = 0$$

Using Eq. (78), Eq. (80) gives

(81) $$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \psi_L \\ \psi_R \end{bmatrix} = 0$$

where R,L mean right-, left-handed. Thus $\psi_R = 0$ and Eq. (79) is

$$\begin{bmatrix} 0 & -i\sigma_K \partial_K + \partial_4 \\ i\sigma_K \partial_K + \partial_4 & 0 \end{bmatrix} \begin{bmatrix} \psi_L \\ 0 \end{bmatrix} = 0$$

or

(82) $$\left[i\sigma_K \partial_K + \partial_4 \right] \psi_L = 0$$

With $\psi_L = u_L e^{ipx}$, Eq. (82) tells

$$(\vec{\sigma} \cdot \vec{p})\, u_L = -|\vec{p}|\, u_L$$

confirming that u_L is indeed left-handed. Take, then, two 2-component neutrinos ν_e and ν_μ, both satisfyng Eq. (82) (i.e. both left-handed)

(83) $$(i\,\sigma_K \partial_K + \partial_4)\,\nu_{\substack{e\\ \mu}} = 0$$

Form the 4-component neutrino

(84) $$\nu = \begin{bmatrix} \nu_e \\ i\,\sigma_2 \nu_\mu^* \end{bmatrix}$$

It can then be shown that ν satisfies a 4-component neutrino equation

(85) $$\gamma \partial \nu = 0$$

In fact Eq. (85) is

$$\begin{bmatrix} 0 & -i\,\sigma_K \partial_K + \partial_4 \\ i\,\sigma_K \partial_K + \partial_4 & 0 \end{bmatrix} \begin{bmatrix} \nu_e \\ i\,\sigma_2 \nu_\mu^* \end{bmatrix} = 0$$

or, equivalently

$$(i\,\sigma_K \partial_K + \partial_4)\,\nu_e = 0$$

and

$$(-i\,\sigma_K \partial_K + \partial_4)\, i\,\sigma_2 \nu_\mu^* = 0$$

The first equation is (83), the second becomes (83) after complex conjugation and multiplication by σ_2. Thus Eq. (85), with Eq. (84), is equivalent to Eq. (83).

Now, what form does lepton conservation (76) take in this representation? The transformations (76) become

$$e \to e^{i\lambda} e \qquad \bar{e} \to \bar{e}\, e^{-i\lambda}$$

$$\mu \to e^{i\lambda} \mu \qquad \bar{\mu} \to \bar{\mu}\, e^{-i\lambda}$$

$$\nu \to e^{i\lambda\gamma_5} \nu \qquad \bar{\nu} \to \bar{\nu}\, e^{-i\lambda\gamma_5}$$

In fact

$$e^{i\lambda\gamma_5} = \begin{bmatrix} e^{i\lambda} & 0 \\ 0 & e^{-i\lambda} \end{bmatrix}$$

and

$$e^{i\lambda\gamma_5}\,\nu = e^{i\lambda\gamma_5}\begin{bmatrix} \nu_e \\ i\sigma_2\nu_\mu^* \end{bmatrix} = \begin{bmatrix} e^{i\lambda}\nu_e \\ i\sigma_2(e^{+i\lambda}\nu_\mu)^* \end{bmatrix}$$

so that $\nu_e \to e^{i\lambda}\nu_e$ and $\nu_\mu \to e^{i\lambda}\nu_\mu$. And next, what about muon number conservation (77)? It is easy to see it becomes

$$(87)\qquad \begin{array}{ll} e \to e^{-im}\, e & \bar{e} \to e^{im}\,\bar{e} \\ \mu \to e^{im}\mu & \bar{\mu} \to e^{-im}\bar{\mu} \\ \nu \to e^{-im}\,\nu & \bar{\nu} \to e^{im}\bar{\nu} \end{array}$$

In fact

$$e^{-im}\,\nu = e^{-im}\begin{bmatrix} \nu_e \\ i\sigma_2\nu_\mu \end{bmatrix} = \begin{bmatrix} e^{-im}\nu_e \\ i\sigma_2(e^{im}\nu_\mu)^* \end{bmatrix}$$

so that $\nu_e \to e^{-im}\nu_e$ and $\nu_\mu \to e^{im}\nu_\mu$.

27. In order to write the Lagrangian in terms of the 4-component ν, we have to invert Eq. (84). To this purpose we first have to find out what the charge conjugate spinor ψ^c of a spinor ψ is, in our representation with γ_5 diagonal. Take a Dirac equation with an external electromagnetic field

$$\left[\gamma(\partial - ieA) + m\right]\psi = 0$$

If now you complex conjugate it and multiply by γ_2 you find

$$\left[\gamma(\partial + ieA) + m\right](\gamma_2\psi^*) = 0$$

The reality conditions for the γ in our particular representation have been used at this step. We thus see that in our representation with γ_5 diagonal

(88) $$\psi^c = \gamma_2\psi^*$$

since such a ψ^c satisfies an equation with charge opposite to that in the equation satisfied by ψ. Now, from Eqs. (84) and (78)

(89)
$$\nu_c = \gamma_2\nu^* = \begin{bmatrix} 0 & -i\sigma_2 \\ i\sigma_2 & 0 \end{bmatrix} \begin{bmatrix} \nu_e^* \\ i\sigma_2\nu_\mu \end{bmatrix} = \begin{bmatrix} \nu_\mu \\ i\sigma_2\nu_e^* \end{bmatrix}$$

and we can now invert Eq. (84) by projecting out the upper components from both ν and ν^c by use of the projection $\frac{1}{2}(1 + \gamma_5) = a$, which applied to a spinor, in our representation, projects out the upper component. Thus

(90) $$\nu_e = a\nu \qquad \nu_\mu = a\nu^c$$

The Lagrangian takes the form

(91)
$$\mathcal{L} = -\bar{\nu}\gamma\partial\nu - \bar{\nu}^c\gamma\partial\nu^c - \bar{e}(\gamma\partial + m_e)e - \bar{\mu}(\gamma\partial + m_\mu)\mu +$$
$$+ G\left[(\bar{e}\gamma a\nu) + (\bar{\mu}\gamma a\nu^c) + \ldots\right]\left[(\bar{\nu}\gamma a e) + (\bar{\nu}^c\gamma a\mu) + \ldots\right] + \text{other terms.}$$

Further Symmetries

28. For Lagrangians such as Eq. (91), with its characteristic appearance of both ν and ν^c, Pauli and Gürsey had considered the group of transformation (the Pauli -

Gürsey group) given by

$$(92)\quad \begin{cases} \nu \to d\nu + b\,\gamma_5\,\nu^c \\ \nu^c \to d^*\nu^c + b^*\gamma_5\,\nu \end{cases} \qquad \begin{cases} \bar{\nu} \to d^*\bar{\nu} - b^*\,\bar{\nu}^c\,\gamma_5 \\ \bar{\nu}^c \to d\,\bar{\nu}^c + b\,\bar{\nu}\gamma_5 \end{cases}$$

with d, b complex numbers satisfying

$$(93)\qquad |d|^2 + |b|^2 = 1$$

The free neutrino Lagrangian is invariant under Eq. (92). In fact, using Eqs. (92) and (93).

$$\bar{\nu}\gamma\partial\nu + \bar{\nu}^c\gamma\partial\nu^c \to (|d|^2 + |b|^2)(\bar{\nu}\gamma\partial\nu + \bar{\nu}^c\gamma\partial\,\nu^c)$$

However the current

$$(94)\qquad \bar{e}\gamma a\,\nu + \bar{\mu}\gamma a\,\nu^c \to (d\bar{e} - b^*\bar{\mu})\,\gamma a\,\nu + (d^*\bar{\mu} + b\bar{e})\,\gamma a\,\nu^c$$

and it is not invariant, unless b = 0. In this case we can write, from Eq. (93)

$$(95)\qquad d = e^{-im},\ b = 0$$

where m is a real number and we have

$$(96)\qquad \bar{e}\gamma a\,\nu + \bar{\mu}\gamma a\,\nu^c \to e^{-im}(\bar{e}\gamma a\,\nu) + e^{im}(\bar{\mu}\gamma a\,\nu^c)$$

If we now "complete" our transformation on the neutrino field, by also acting on e and μ with

$$(97)\qquad \begin{aligned} e &\to e^{-im}e \\ \mu &\to e^{im}\mu \end{aligned}$$

we can reabsorb the phase factors in Eq. (96) and get full invariance. But Eq. (97), together with Eq. (95) inserted into Eq. (92), taken altogether are nothing else than Eq. (87). Therefore this subgroup of the Pauli-Gürsey group is nothing else than our gauge group (87) expressing muon conservation.

29. Equation (94) also teaches you that you might have invariance of the current under the full group provided you extend the transformation also to e and μ (10), by transforming according to

(98)
$$d\bar{e} - b^{*}\bar{\mu} \rightarrow \bar{e}$$
$$d^{*}\bar{\mu} + b\,\bar{e} \rightarrow \bar{\mu}$$

In matrix notations this is

(99)
$$U \begin{bmatrix} e \\ \mu \end{bmatrix} = \begin{bmatrix} d^{*} & -b \\ b^{*} & d \end{bmatrix} \begin{bmatrix} e \\ \mu \end{bmatrix} \rightarrow \begin{bmatrix} e \\ \mu \end{bmatrix}$$

where U is a unitary matrix

(100)
$$U^{\dagger} U = 1$$

with determinant unity

(101)
$$\det U = |d|^2 + |b|^2 = 1$$

The matrices U satisfying Eqs. (100) and (101) form the subgroup SU(2) of the unitary group in two dimensions U(2). However, the μ-e mass difference does not allow Eq. (99) to be a symmetry operation for the whole Lagrangian. This is a situation already met with various times in theoretical physics. We also note that in terms of $\nu_e = a\nu$ and $\nu_\mu = a\,\nu^{c}$ the Pauli-Gürsey transformation is given by

(102)
$$\nu_e = a\nu \rightarrow d(a\nu) + b\,(a\,\nu^{c}) = d\,\nu_e + b\,\nu_\mu$$
$$\nu_\mu = a\,\nu^{c} \rightarrow d^{*}(a\,\nu^{c}) - b^{*}(a\nu) = d^{*}\nu_\mu - b^{*}\nu_e$$

or in doublet notation

$$\begin{bmatrix} \nu_e \\ \nu_\mu \end{bmatrix} \rightarrow \begin{bmatrix} d & b \\ -b^{*} & d^{*} \end{bmatrix} \begin{bmatrix} \nu_e \\ \nu_\mu \end{bmatrix} = U^{-1} \begin{bmatrix} \nu_e \\ \nu_\mu \end{bmatrix}$$

Therefore, in this description, the transformation on e, μ, and that on ν_e, ν_μ are both realized by a unitary unimodular matrix U. Lepton conservation together with this group SU(2) give as a direct product the full unitary group U(2).

30. I shall add, in this section, a summary of some further speculations on higher symmetry schemes for leptons, in which I have been involved lastly. I shall adopt the four-component description of the neutrino. I shall also define as leptons: the positive muon, the neutrino, and the negative electron. The negative muon, the antineutrino, and the positiveelectron are antileptons. Furthermore lepton number is conserved (additively). It follows from lepton conservation and from our definition of leptons that processes such as:

$$\mu \rightarrow e + \gamma$$

$$\mu \rightarrow 3e$$

$$\mu^- + \text{nucleus} \rightarrow e^- + \text{nucleus}$$

$$e^- + \mu^+ \rightarrow \mu^- + e^+$$

$$e^- + e^- \rightarrow \mu^- + \mu^+$$

are forbidden, since they would imply a change of lepton number (for a multiplicative lepton number rule the last two processes would be allowed). It also follows directly that double β-decay is forbidden, that a neutrino - antineutrino pair is emitted in μ decay, etc. Our leptonic world contains three basic leptons: the muon, the neutrino, and the electron. It is then natural to try a classification of the leptons and of the leptonic currents according to their behaviour under the group U_3 of unitary transformations on three variables and under its subgroups. The group U_3 can be split into $U_1 \times SU_3$, where SU_3 is the unimodular group in three dimensions and U_1 can be represented by the phase transformation corresponding to the lepton gauge. Before sketching how the argument runs we summarize the main results of such a classification:

1 - Leptonic currents: In Tables IIa and IIb we report the possible independent sets of currents. They divide into two groups. The currents of the sets of the first group have a definite behaviour under parity. Sets of the first group must be excluded because they would not allow for parity non-conservation in muon decay. The sets of the second group have the chiral character of the AV theory. The charged current

$$\tfrac{1}{2}(j_1 - ij_2) = -\tfrac{1}{2}(\pm \bar{e}\gamma a \nu + \bar{\nu}\gamma \bar{a} \mu)$$

for such sets is also equal to

$$(103) \qquad = -\tfrac{1}{2}(\pm\, \bar{e}\, \gamma\, \nu_e - \mu^c \gamma\, \nu_\mu)$$

with the position (90). One sees that (103) is the well-know expression for the charged lepton current (with $\Delta Q = 1$). We conclude that only sets of the second group can be physically acceptable. Furthermore we are led to choose for a further classification that subgroup SU_2, whose generators are space integrals of the fourth components of $\frac{1}{2}(j_1 \pm i j_2)$ and j_3.

2 - Lepton classification: The choice, from experiment, of sets of the second group implies that the positive-helicity leptons (that we call: μ_+, ν_- and e_+) and the negative-helicity leptons (that we call: μ_-, ν_+ and e_-) transform, under SU_3, according to inequivalent three-dimensional representations. This is illustrated in fig. 2 where the weight-diagrams of the representations $D^3(1,0)$ and $D^3(0,1)$ are reported. We have chosen the positive-helicity leptons to transform according to $D^3(1,0)$ and the negative-helicity leptons to transform according to $D^3(0,1)$.

3 - Lepton-isospin and lepton-strangeness: in the weight diagrams of fig. 2, $(F_3^{(+)}, F_8^{(+)})$ and $(F_3^{(-)}, F_8^{(-)})$ are commuting group generators. We introduce

$$(104) \qquad I_3^{(+)} = F_3^{(+)}$$

$$(104') \qquad S^{(+)} = 2\sqrt{3}\, F_8^{(+)} - L$$

(where L is the lepton number) and, similarly, $I_3^{(-)}$ and $S^{(-)}$. We can write for the charge Q:

$$Q = I_3^{(+)} + \frac{L+S^{(+)}}{2} = I_3^{(-)} + \frac{L+S^{(-)}}{2}$$

and we can label particles and currents by their right-isospin $\vec{I}^{(+)}$, left-isospin $\vec{I}^{(-)}$, right-strangeness $S^{(+)}$ and left-changeness $S^{(-)}$ (see tables IIIa, IIIb, and IV).

4 - Baryon-lepton symmetry: in the last columns of the tables IIIa, IIIb, and IV we have reported the "corresponding baryon" and the "corresponding meson" for each particle and current, i.e. the baryon or boson with

corresponding quantum numbers (the correspondence is: lepton number $\leftrightarrow$ nucleon number; $Q \leftrightarrow Q$; $S^{(+)}$, $S^{(-)} \leftrightarrow S$; $I^{(+)}$, $I^{(-)} \leftrightarrow I$). The baryons Z^- (isotopic spin $I = 0$, $S = -3$) and X^+ ($I = 0$, $S = +1$), and the mesons ν ($I = \frac{1}{2}$, $S = \pm 3$) have not been found so far. The baryon-lepton correspondence rules of tables IIIa, IIIb and IV replace the Gamba, Marshak, Okubo correspondence rule $p \leftrightarrow \nu$, $n \leftrightarrow e^-$ $\Lambda \leftrightarrow \mu^-$ (11).

5 - Weak four-lepton coupling: invariance of the weak four-lepton coupling under the full unitary group can be excluded: it would lead to a parity conserving interaction. We generate the weak four-lepton Lagrangian, L' by self-coupling of the currents of table IIb (for each of the chosen sets of the second group). We write

(105) $$L' = gL_1 + fL_2 + ihL_3$$

where L_1, L_2 and L_3 are invariant under the lepton-isospin subgroup SU_2. L_1 arises from the selfcoupling of j_1, j_2, j_3; L_2 from the self-coupling of j_4, j_5, j_6, j_7; and L_3 from the self-coupling of j_8. In the limit of unitary symmetry $g = f = h$. From the measured value of the ξ parameter in μ-decay we can show that

(106) $$f < 0.2g$$

We shall now sketch briefly the main lines of the argument. The generators of the unitary group are called $F_o = L$, F_1, F_2, ... F_8, and are assumed to be integrals of partially-conserved local currents

(107) $$F_i = -i \int j_i(x) d\sigma_\mu$$

The current j_i, because of its vector character, can be decomposed into a contribution from positive-helicity particles and one from negative-helicity particles. Correspondingly

(108) $$F_i = F_i^{(+)} + F_i^{(-)}$$

and $F_i^{(+)}$, $F_i^{(-)}$ satisfy the commutation relations

(109) $$\left[F_i^{(+)}, F_j^{(+)} \right] = i\, f_{ijk} F_k^{(+)}$$

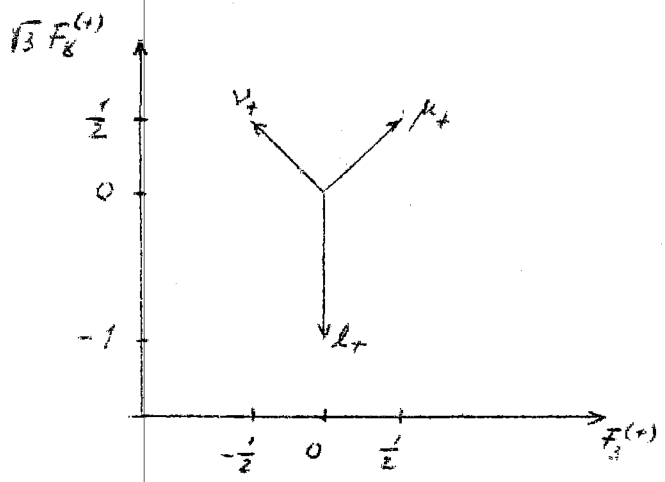


Fig. 2a - Weight diagram for the positive helicity leptons.

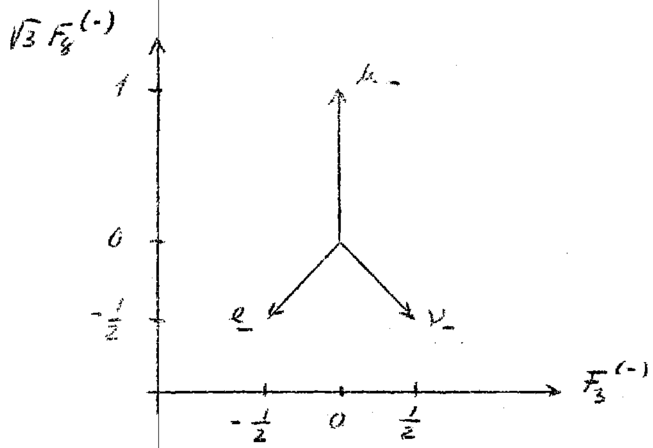


Fig. 2b - Weight diagram for the negative helicity leptons.

$\frac{1}{2}(j_1+ij_2)$	$-\frac{i}{2}\bar{\mu}\gamma\nu$	$-\frac{i}{2}\bar{\mu}\gamma\nu$	$\frac{i}{2}\bar{\mu}\gamma\gamma_5\nu$	$\frac{i}{2}\bar{\mu}\gamma\gamma_5\nu$
j_3	$-\frac{i}{2}(\bar{\mu}\gamma\mu-\bar{\nu}\gamma\nu)$			
$\frac{1}{2}(j_4+ij_5)$	$-\frac{i}{2}\bar{\mu}\gamma e$	$\frac{i}{2}\bar{\mu}\gamma\gamma_5 e$	$\frac{i}{2}\bar{\mu}\gamma\gamma_5 e$	$-\frac{i}{2}\bar{\mu}\gamma e$
$\frac{1}{2}(j_6+ij_7)$	$-\frac{i}{2}\bar{\nu}\gamma e$	$\frac{i}{2}\bar{\nu}\gamma\gamma_5 e$	$-\frac{i}{2}\bar{\nu}\gamma e$	$\frac{i}{2}\bar{\nu}\gamma\gamma_5 e$
j_8	$-\frac{i}{2}\frac{1}{\sqrt{3}}(\bar{\mu}\gamma\mu+\bar{\nu}\gamma\nu-2\bar{e}\gamma e)$			

Table IIa - Sets of the first group:positive helicity leptons and negative helicity leptons tran sform according to equivalent representations of SU_3.-

$\frac{1}{2}(j_1+ij_2)$	$-\frac{i}{2}(\bar{\nu}\gamma a e+\bar{\mu}\gamma\bar{a}\nu)$	$-\frac{i}{2}(\bar{\nu}\gamma a e+\bar{\mu}\gamma\bar{a}\nu)$	$-\frac{i}{2}(-\bar{\nu}\gamma a e+\bar{\mu}\gamma\bar{a}\nu)$	$-\frac{i}{2}(-\bar{\nu}\gamma a e-\bar{\mu}\gamma\bar{a}\nu)$
j_3	$-\frac{i}{2}(\bar{\nu}\gamma a\nu-\bar{e}\gamma a e+\bar{\mu}\gamma\bar{a}\mu-\bar{\nu}\gamma\bar{a}\nu)$			
$\frac{1}{2}(j_4+ij_5)$	$-\frac{i}{2}(-\bar{\mu}\gamma a e+\bar{\mu}\gamma\bar{a}e)$	$-\frac{i}{2}(\bar{\mu}\gamma a e+\bar{\mu}\gamma\bar{a}e)$	$-\frac{i}{2}(\bar{\mu}\gamma a e-\bar{\mu}\gamma\bar{a}e)$	$-\frac{i}{2}(-\bar{\mu}\gamma a e+\bar{\mu}\gamma\bar{a}e)$
$\frac{1}{2}(j_6+ij_7)$	$-\frac{i}{2}(\bar{\mu}\gamma a\nu+\bar{\nu}\gamma\bar{a}e)$	$-\frac{i}{2}(-\bar{\mu}\gamma a\nu+\bar{\nu}\gamma\bar{a}e)$	$-\frac{i}{2}(\bar{\mu}\gamma a\nu+\bar{\nu}\gamma\bar{a}e$	$-\frac{i}{2}(-\bar{\mu}\gamma a\nu+\bar{\nu}\gamma\bar{a}e)$
j_8	$-\frac{i}{2}\frac{1}{\sqrt{3}}(2\bar{\mu}\gamma a\mu-\bar{\nu}\gamma a\nu-\bar{e}\gamma a e+\bar{\mu}\gamma\bar{a}\mu+\bar{\nu}\gamma\bar{a}\nu-2\bar{e}\gamma\bar{a}e)$			

Table IIb - Sets of the second group: positive helicity leptons and negative helicity leptons tran sform according to inequivalent representations of SU_3.-

particle	lepton number L	charge Q	$S^{(+)}$	$\lvert\vec{I}^{(+)}\rvert$	$I_3^{(+)}$	Corresponding baryon
μ_+	$+1$	$+1$	0	$\frac{1}{2}$	$\frac{1}{2}$	p
ν_+	$+1$	0	0		$-\frac{1}{2}$	n
e_+	$+1$	-1	-3	0	0	Z^-

Table IIIa - Quantum number assignments to the leptons with positive helicity. The corresponding baryons are indicated in last column. The baryon Z^- has not yet been discovered.

particle	lepton number L	charge Q	$S^{(-)}$	$\lvert\vec{I}^{(-)}\rvert$	$I_3^{(-)}$	Corresponding baryon
μ_-	$+1$	$+1$	1	0	0	X^+
ν_-	$+1$	0	-2	$\frac{1}{2}$	$\frac{1}{2}$	Ξ^0
e_-	$+1$	-1	-2		$-\frac{1}{2}$	Ξ^-

Table IIIb - Quantum number assignments to the leptons with negative helicity. The corresponding baryons are indicated in the last column. The baryon X^+ has not yet been discovered.

current	lepton number L	charge Q	$S^{(-)} = S^{(+)}$	$\lvert\vec{I}^{(-)}\rvert - \lvert\vec{I}^{(+)}\rvert$	$I_3^{(-)} = I_3^{(+)}$	corresponding meson
$\frac{1}{2}(j_1 \pm i j_2)$	0	± 1	0	1	± 1	ρ^+, ρ^-
j_3	0	0	0	1	0	ρ^0
$\frac{1}{2}(j_4 \pm i j_5)$	0	± 2	± 3	$\frac{1}{2}$	$\pm \frac{1}{2}$	$\varphi^{++}, \varphi^{--}$
$\frac{1}{2}(j_6 \pm i j_7)$	0	± 1	± 3	$\frac{1}{2}$	$\pm \frac{1}{2}$	φ^{+}, φ^{-}
j_8	0	0	0	0	0	ω^0

Table IV - Quantum number assignments to the currents. The corresponding (vector) mesons are indicated in the last column. The mesons φ have not yet been discovered.-

$$(109') \qquad \left[F_i^{(-)}, F_j^{(-)} \right] = iF_{ijk} F_k^{(-)}$$

The commutation relations with the charge operator Q must be of the form

$$(110) \qquad \left[Q, F_i^{(+)} \right] = C_{ik} F_k^{(+)}$$

$$(111) \qquad \left[Q, F_i^{(-)} \right] = C_{ik} F_k^{(-)}$$

We do not need to specify C_{ik}.

We now construct the 3-dimensional representations f_i of F_i. We distinguish two cases:

(I) $f_i^{(-)}$ and $f_i^{(+)}$ are related by a similarity transformation

$$(112) \qquad f_i^{(-)} = w f_i^{(+)} w^{-1}$$

(II) $f_i^{(-)}$ and $f_i^{(+)}$ are not related by a similarity transformation. In such a case

$$(113) \qquad f_i^{(-)} = w \tilde{f}_i^{(+)} w^{-1}$$

must hold, where $\tilde{f}_i^{(+)}$ is the representations contragradient to $f_i^{(+)}$. Using (110) and (111) we can show that: in case (I)

$$(114) \qquad \left[q, w \right] = 0$$

while in case (II)

$$(115) \qquad \left\{ q, w \right\} = 0$$

where $\{\ldots\}$ denotes the anticommutater and q is the 3x3 representation of Q. In both cases, (114) or (115) define w in terms of two real parameters. Having constructed $f_i^{(-)}$ we demand for it the same reality properties of $f_i^{(+)}$ (time reversal invariance). Such a procedure leads directly to the currents of Tables IIa and IIb.

The Lagrangians L_1, L_3 are given by

$$L_1 = \pm (\bar{\nu}\gamma ae)(\bar{\nu}\gamma\bar{a}\mu) \pm (\bar{\mu}\gamma\bar{a}\nu)(\bar{e}\gamma a\nu) +$$

$$+ \frac{1}{4}(\bar{\nu}\gamma\nu)(\bar{e}\gamma ae) + \frac{1}{4}(\bar{\nu}\gamma\nu)(\bar{\mu}\gamma\bar{a}\mu) +$$

(116) $$- \tfrac{1}{2}(\bar{e}\gamma ae)(\bar{\mu}\gamma\bar{a}\mu) +$$

$$+ \frac{1}{4}(\bar{e}\gamma ae)(\bar{e}\gamma ae) + \frac{1}{4}(\bar{\mu}\gamma a\mu)(\bar{\mu}\gamma\bar{a}\mu) +$$

$$+ \frac{1}{4}(\bar{\nu}\gamma\gamma_5\nu)(\bar{\nu}\gamma\gamma_5\nu)$$

$$L_3 = \frac{1}{12}\Big[2(\bar{\mu}\gamma a\mu) + (\bar{\mu}\gamma a\mu)\Big]\Big[2(\bar{\mu}\gamma a\mu) +$$

$$+ (\bar{\mu}\gamma\bar{a}\mu)\Big] + \frac{1}{12}\Big[2(\bar{e}\gamma\bar{a}e) +$$

$$+ (\bar{e}\gamma ae)\Big]\Big[2(\bar{e}\gamma\bar{a}e) + (\bar{e}\gamma ae)\Big] +$$

$$+ \frac{1}{12}(\bar{\nu}\gamma\gamma_5\nu)(\bar{\nu}\gamma\gamma_5\nu) + \frac{1}{6}\Big[2(\bar{e}\gamma\bar{a}e) +$$

$$+ (\bar{e}\gamma ae)\Big](\bar{\nu}\gamma\gamma_5\nu) - \frac{1}{6}\Big[2(\bar{\mu}\gamma a\mu) +$$

$$+ (\bar{\mu}\gamma\bar{a}\mu)\Big](\bar{\nu}\gamma\gamma_5\nu) - \frac{1}{6}\Big[2(\bar{\mu}\gamma a\mu) +$$

$$+ (\bar{\mu}\gamma\bar{a}\mu)\Big]\Big[2(\bar{e}\gamma\bar{a}e) + (\bar{e}\gamma ae)\Big]$$

We do not report the form of L_2, since as we have indicated it is presumably absent. Both L_1 and L_2 would contribute to μ-decay. The total contribution would be of the form

(118) $$(\bar{e}\gamma(q+p\gamma_5)\mu^c)(\bar{\nu}^c\gamma\gamma_5\nu)$$

If L_2 is absent, $p = q$ and (118) becomes the known μ -decay Lagrangian. The physical consequences of (118) can be simply read off from paper on Pauli-Pursey invariants in μ-decay [10]. It gives for the muons decay parameters $\rho = \frac{3}{4}$, $\delta = \frac{3}{4}$, and $\xi = -2pq/(p^2 + q^2)$. Using Steinberger's figures [13] for ξ we have obtained (106).

The result (106) indicates that the self-coupling of j_4, j_5, j_6, j_7 is presumably absent. Such coupling, if mediated by vector bosons, would have required double charged bosons. The coupling of j_6 and j_7 to strong interacting currents (both strangeness conserving and strangeness non- conserving) seems to be experimentally excluded.

Invariance under SU_2 of the weak four-lepton interaction can directly be checked by measurement of the cross-section for scattering of ν_e on e (with neutrinos from nuclear reactors). It also allows to predict scattering of ν_μ on e and weak effects in scattering and μ-pair production in high energy electron-positron colliding beams.

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Appendix I

Differential cross-section in the laboratory system for $\bar{\nu} + p \rightarrow \ell^+ + n$ derived from formulae (4), (3), (6) and (8) of the text:

$$\sigma(\theta)\, d(\cos\theta) = \frac{G^2}{\pi} \mathcal{E}_\nu^2\, \Sigma(\theta, \mathcal{E}_\nu)\, d(\cos\theta)$$

where $\mathcal{E}_\nu$ = laboratory neutrino energy, θ = laboratory scattering angle, and

$$\Sigma(\theta, \mathcal{E}_\nu) = \left(1 + 2\xi \sin^2\frac{\theta}{2}\right)^{-3} \Bigg\{ \cos^2\frac{\theta}{2}\left[F_1^{\,2} + \right.$$

$$\left. + \frac{K^2}{4M^2}\left(2(F_1 + \mu F_2)^2 \operatorname{tg}^2\frac{\theta}{2} + \mu^2 F_2^{\,2}\right)\right] +$$

$$+ H_1^{\,2}\left[1 + \sin^2\frac{\theta}{2} + 2\xi^2 \frac{\sin^4\frac{\theta}{2}}{1 + 2\xi \sin^2\frac{\theta}{2}}\right] +$$

$$- H_1(F_1 + \mu F_2)\, 4M^2 \xi \frac{\sin^4\frac{\theta}{2}}{1 + 2\xi\sin^2\frac{\theta}{2}} \left(1 + \right.$$

$$\left. + \sin^2\frac{\theta}{2}\right)\Bigg\}$$

with $\xi = \mathcal{E}_\nu / M$. For $\nu + n \rightarrow p + \ell^-$ the A-V interference term [that proportional to $H_1(F_1 + \mu F_2)$] changes sign, and the rest is identical.

Appendix II

Write

$$A + B\gamma_5 = (A + B)a + (A - B)\bar{a}$$
$$C + iD\gamma_5 = (C + iD)a + (C - iD)\bar{a}.$$

Now substitute into Eq. (58) together with

$$\psi' = T\psi = (aR + \bar{a}S)$$
$$\bar{\psi}' = (\overline{T\psi}) = \bar{\psi}(\bar{a}R^{\dagger} + aS^{\dagger}).$$

Comparing with Eq. (52) we find

$$aR^{\dagger}(A + B)R + \bar{a}S^{\dagger}(A - B)S = 1 = a + \bar{a}$$
$$\bar{a}R^{\dagger}(C - iD)S + \bar{a}S^{\dagger}(C + iD)R = M = M(a + \bar{a})$$

that are equivalent to Eqs. (61), (61') and (61").

Appendix III

Write

$$T^{-1}\sigma_1 T = aX + \bar{a}Y$$

Then, because of Eq. (68)

$$a\psi = a(aX + \bar{a}Y)\psi = aX\psi$$

Therefore X = 1, Y arbitrary; or

$$T^{-1}\ \sigma_1\ T = a + \bar{a}Y$$

$$\sigma_1 T = Ta + T\bar{a}Y$$

Inserting Eq. (59)

$$a\,\sigma_1 R + \bar{a}\,\sigma_1 S = aR + \bar{a}SY$$

implying the condition

$$\sigma_1 R = R$$

Then take Eq. (61) and what you get by eliminating S from Eqs. (61') and (61"), namely multiplying according to

[hermitian conjugate of Eq. (61')] × [inverse of Eq. (61")] ×
× [Eq. (61")]

which reads

$$R^{\dagger}(C^2 + D^2)(A - B)^{-1}R = M^2$$

Write

$$A + B = a\ P_+ + b\ P_-$$

$$(C^2 + D^2)(A - B)^{-1} = p\ P_+ + q\ P_-$$

Then Eq. (61) and this last equation give

$$R^{\dagger}\ (a\ P_{+} + b\ P_{-})R = a\ R^{\dagger}\ P_{+}R = 1$$
$$R^{\dagger}\ (p\ P_{+} + q\ P_{-})R = p\ R^{\dagger} P_{+}R = M^2$$

where we have used the above condition $\sigma_1 R = R$ to obtain

$$R^{\dagger}\ P_{-}R = R^{\dagger}\ \sigma_1 P_{-}R = R^{\dagger}\tfrac{1}{2}(1 - \sigma_1)R = -R^{\dagger}\ P_{-}R = 0$$

But the equation

$$a\ M^2 = p$$

is incompatible because a and p and numbers, while M^2 contains a term proportional to σ_3 (it is compatible only if $m_{\mu} = m_e$).

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